

A table of boundary slopes of Montesinos knots

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Abstract

This note corrects errors in Hatcher and Oertel's table of boundary slopes of Montesinos knots which have projections with 10 or fewer crossings.

1 Introduction

In [HO], Hatcher and Oertel gave an algorithm for computing the boundary slopes of a Montesinos knot. At the end of their paper they provided a table giving the boundary slopes for each Montesinos knot in the standard table in Rolfsen's book [Rol]. Unfortunately, their table contains several (17) errors. These errors were due to problems with the computer program that generated the table, as well as transcription/printing errors. This note provides a corrected table which was generated by a completely new computer program. Section 2 contains the corrected table. Section 3 describes the precautions I took in writing the new program. Unfortunately, not all mistakes in [HO] were found in the original 1999 version of this note, and this 2010 revision corrects several more (details are given in Section 3). Section 4 describes where the reader can download the new program.

I thank Allen Hatcher and especially Ulrich Oertel for their help with this project. I also thank Thomas Mattman and Marc Culler their help with finding and correcting the additional errors.

2 Revised table

Throughout this section * denotes an entry which differs from the corresponding entry in the table in [HO], and † denotes one that differs from the 1999 version of this note.

Below are the boundary slopes for those Montesinos knots of ten or fewer crossings. Notation follows [HO] and [Rol].

$3_1 = K(1/3) : 0, 6$
$4_1 = K(2/5) : -4, 0, 4$
$5_1 = K(1/5) : 0, 10$
$5_2 = K(3/7) : 0, 4, 10$
$6_1 = K(4/9) : -4, 0, 8$
$6_2 = K(4/11) : -4, 0, 2, 8$
$6_3 = K(5/13) : -6, -2, 0, 2, 6$
$7_1 = K(1/7) : 0, 14$
$7_2 = K(5/11) : 0, 4, 14$
$7_3 = K(4/13) : -14, -8, 0$
$7_4 = K(4/15) : -14, -8, 0$
$7_5 = K(7/17) : 0, 4, 6, 10, 14$
$7_6 = K(7/19) : -4, 0, 4, 6, 10$
$7_7 = K(8/21) : -8, -4, 0, 6$
$8_1 = K(6/13) : -4, 0, 12$
$8_2 = K(6/17) : -4, 0, 6, 12$
$8_3 = K(4/17) : -8, 0, 8$
$8_4 = K(5/19) : -8, -2, 0, 8$
$8_5 = K(1/3, 1/3, 1/2) : -4, 0, 2, 8, 10, 12$
$8_6 = K(10/23) : -4, 0, 2, 6, 12$
$8_7 = K(9/23) : -10, -6, -2, 0, 6$
$8_8 = K(9/25) : -10, -6, -4, 0, 2, 6$
$8_9 = K(7/25) : -8, -2, 0, 2, 8$
$8_{10} = K(1/3, 2/3, 1/2) : -6, -2, 0, 6, 8, 10$
$8_{11} = K(10/27) : -4, 0, 6, 12$
$8_{12} = K(12/29) : -8, -4, 0, 4, 8$
$8_{13} = K(11/29) : -10, -6, -4, -2, 0, 6$
$8_{14}^* = K(12/31) : -4, 0, 4, 6, 8, 12$
$8_{15} = K(2/3, 2/3, 1/2) : -16, -12, -10, -8, -4, -2, 0$
$8_{19} = K(1/3, 1/3, -1/2) : 0, 12$
$8_{20} = K(1/3, 2/3, -1/2) : -10, 0, 8/3$
$8_{21} = K(2/3, 2/3, -1/2) : -12, -6, -2, 0, 1$
$9_1 = K(1/9) : 0, 18$
$9_2 = K(7/15) : 0, 4, 18$
$9_3 = K(6/19) : -18, -12, 0$
$9_4 = K(5/21) : 0, 8, 18$
$9_5 = K(6/23) : -18, -12, -8, 0$
$9_6 = K(11/27) : 0, 4, 10, 14, 18$
$9_7 = K(13/29) : 0, 4, 6, 10, 18$
$9_8 = K(11/31) : -8, -4, 0, 4, 6, 10$
$9_9 = K(9/31) : 0, 6, 8, 14, 18$
$9_{10} = K(10/33) : -18, -12, -6, 0$
$9_{11} = K(14/33) : -14, -10, -4, 0, 4$
$9_{12} = K(13/35) : -4, 0, 6, 8, 14$
$9_{13} = K(10/37) : -18, -14, -12, -8, -6, 0$
$9_{14} = K(14/37) : -12, -8, -4, 0, 6$
$9_{15} = K(16/39) : -14, -10, -8, -4, 0, 4$
$9_{16} = K(1/3, 1/3, 3/2) : 0, 4, 6, 10, 12, 14, 16, 18$
$9_{17} = K(14/39) : -8, -4, -2, 0, 4, 10$
$9_{18} = K(17/41) : 0, 4, 8, 10, 12, 14, 18$
$9_{19} = K(16/41) : -8, -4, 0, 4, 10$
$9_{20} = K(15/41) : -4, 0, 2, 6, 8, 14$
$9_{21} = K(18/43) : -14, -10, -8, -4, 0, 4$
$9_{22} = K(3/5, 1/3, 1/2) : -8, -4, -2, 0, 2, 4, 6, 8, 10$
$9_{23} = K(19/45) : 0, 4, 8, 10, 14, 18$
$9_{24} = K(1/3, 2/3, 3/2) : -10, -6, -4, 0, 2, 4, 6, 8$
$9_{25} = K(2/5, 2/3, 1/2) : -14, -10, -8, -6, -4, -2, 0, 2, 4$

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$9_{26} = K(18/47) : -12, -8, -6, -4, 0, 6$
 $9_{27} = K(19/49) : -8, -4, -2, 0, 2, 4, 6, 10$
 $9_{28} = K(2/3, 2/3, 3/2) : -12, -8, -6, -2, 0, 2, 4, 6$
 $9_{30} = K(3/5, 2/3, 1/2) : -10, -6, -4, -2, 0, 2, 4, 6, 8$
 $9_{31} = K(21/55) : -6, -2, 0, 2, 6, 12$
 $9_{35} = K(1/3, 1/3, 1/3) : -18, -12, -4, 0$
 $9_{36} = K(2/5, 1/3, 1/2) : -4, 0, 2, 4, 6, 8, 10, 12, 14$
 $9_{37} = K(1/3, 2/3, 2/3) : -10, -4, 0, 4, 8$
 $9_{42} = K(2/5, 1/3, -1/2) : -8, 0, 8/3, 6$
 $9_{43} = K(3/5, 1/3, -1/2) : -4, 0, 6, 8, 32/3$
 $9_{44} = K(2/5, 2/3, -1/2) : -10, -2, 0, 1, 2, 14/3$
 $9_{45} = K(3/5, 2/3, -1/2) : -14, -10, -8, -4, -2, 0, 1$
 $9_{46} = K(1/3, 1/3, -1/3) : -12, 0, 2$
 $9_{48} = K(2/3, 2/3, -1/3) : -4, 0, 4, 8, 11$
 $10_1 = K(8/17) : -4, 0, 16$
 $10_2 = K(8/23) : -4, 0, 10, 16$
 $10_3 = K(6/25) : -8, 0, 12$
 $10_4 = K(7/27) : -12, -6, 0, 8$
 $10_5 = K(13/33) : -14, -10, -6, 0, 6$
 $10_6 = K(16/37) : -4, 0, 6, 10, 16$
 $10_7 = K(16/43) : -4, 0, 6, 10, 16$
 $10_8 = K(6/29) : -8, 0, 2, 12$
 $10_9 = K(11/39) : -12, -6, -2, 0, 8$
 $10_{10} = K(17/45) : -14, -10, -6, -4, 0, 6$
 $10_{11} = K(13/43) : -8, -2, 0, 6, 12$
 $10_{12} = K(17/47) : -14, -10, -8, -2, 0, 6$
 $10_{13} = K(22/53) : -8, -4, 0, 4, 8, 12$
 $10_{14} = K(22/57) : -4, 0, 4, 8, 10, 12, 16$
 $10_{15} = K(19/43) : -10, -6, 0, 2, 4, 10$
 $10_{16} = K(14/47) : -12, -6, -2, 0, 4, 8$
 $10_{17} = K(9/41) : -10, -2, 0, 2, 10$
 $10_{18} = K(23/55) : -8, -4, 0, 2, 4, 8, 12$
 $10_{19} = K(14/51) : -10, -4, -2, 0, 2, 4, 10$
 $10_{20} = K(16/35) : -4, 0, 2, 6, 16$
 $10_{21} = K(16/45) : -4, 0, 4, 10, 16$
 $10_{22} = K(13/49) : -12, -6, 0, 2, 8$
 $10_{23} = K(23/59) : -14, -10, -8, -6, -4, 0, 6$
 $10_{24} = K(24/55) : -4, 0, 4, 6, 10, 16$
 $10_{25} = K(24/65) : -4, 0, 2, 6, 10, 12, 16$
 $10_{26} = K(17/61) : -12, -6, -2, 0, 4, 8$
 $10_{27} = K(27/71) : -14, -10, -6, -4, 0, 6$
 $10_{28} = K(19/53) : -14, -10, -8, -4, -2, 0, 6$
 $10_{29} = K(26/63) : -8, -4, -2, 0, 2, 4, 8, 12$
 $10_{30} = K(26/67) : -4, 0, 4, 8, 10, 12, 16$
 $10_{31} = K(25/57) : -10, -6, -4, 0, 2, 4, 10$
 $10_{32} = K(29/69) : -8, -4, -2, 0, 2, 4, 6, 8, 12$
 $10_{33} = K(18/65) : -10, -4, -2, 0, 2, 4, 10$
 $10_{34} = K(13/37) : -14, -10, -4, 0, 2, 6$
 $10_{35} = K(20/49) : -12, -8, -4, 0, 4, 8$
 $10_{36} = K(20/51) : -4, 0, 4, 8, 10, 16$
 $10_{37} = K(23/53) : -10, -6, -4, 0, 4, 6, 10$
 $10_{38} = K(25/59) : -4, 0, 4, 6, 8, 10, 12, 16$
 $10_{39} = K(22/61) : -4, 0, 2, 4, 6, 8, 10, 12, 16$
 $10_{40} = K(29/75) : -14, -10, -6, -4, -2, 0, 2, 6$
 $10_{41} = K(26/71) : -8, -4, 0, 2, 4, 6, 8, 12$
 $10_{42} = K(31/81) : -10, -6, -4, -2, 0, 2, 4, 6, 10$
 $10_{43} = K(27/73) : -10, -6, -4, 0, 4, 6, 10$

$10_{44} = K(30/79) : -8, -4, 0, 2, 4, 6, 12$
 $10_{45} = K(34/89) : -10, -6, -4, -2, 0, 2, 4, 6, 10$
 $10_{46} = K(1/5, 1/3, 1/2) : -4, 0, 2, 6, 8, 12, 14, 16$
 $10_{47} = K(1/5, 2/3, 1/2) : -6, -2, 0, 4, 6, 10, 12, 14$
 $10_{48} = K(4/5, 1/3, 1/2) : -10, -6, -4, -2, 0, 2, 6, 8, 10$
 $10_{49} = K(4/5, 2/3, 1/2) : -20, -16, -14, -12, -10, -8, -4, -2, 0$
 $10_{50} = K(3/7, 1/3, 1/2) : -4, 0, 2, 6, 8, 10, 12, 14, 16$
 $10_{51} = K(3/7, 2/3, 1/2) : -6, -2, 0, 4, 6, 8, 10, 12, 14$
 $10_{52} = K(4/7, 1/3, 1/2) : -10, -6, -4, 0, 2, 6, 8, 10$
 $10_{53} = K(4/7, 2/3, 1/2) : -20, -16, -14, -12, -10, -8, -4, -2, 0$
 $10_{54} = K(2/7, 1/3, 1/2) : -10, -6, -4, 0, 2, 4, 6, 8, 10$
 $10_{55} = K(2/7, 2/3, 1/2) : -20, -16, -14, -10, -8, -6, -4, -2, 0$
 $10_{56} = K(5/7, 1/3, 1/2) : -4, 0, 2, 4, 6, 8, 12, 14, 16$
 $10_{57} = K(5/7, 2/3, 1/2) : -6, -2, 0, 2, 4, 6, 10, 12, 14$
 $10_{58} = K(2/5, 2/5, 1/2) : -12, -8, -4, -2, 0, 2, 4, 6, 8$
 $10_{59} = K(2/5, 3/5, 1/2) : -8, -4, 0, 2, 4, 6, 8, 10, 12$
 $10_{60}^* = K(3/5, 3/5, 1/2) : -12, -8, -4, 0, 2, 6, 8$
 $10_{61} = K(1/4, 1/3, 1/3) : -8, -2, 0, 6, 8, 12$
 $10_{62} = K(1/4, 1/3, 2/3) : -6, 0, 2, 8, 10, 14$
 $10_{63} = K(1/4, 2/3, 2/3) : -20, -14, -12, -8, -6, -4, 0$
 $10_{64}^{*\dagger} = K(3/4, 1/3, 1/3) : -8, -2, 0, 2, 4, 6, 8, 12$
 $10_{65} = K(3/4, 1/3, 2/3) : -6, 0, 2, 4, 8, 10, 14$
 $10_{66} = K(3/4, 2/3, 2/3) : -20, -14, -12, -10, -8, -6, -4, 0$
 $10_{67} = K(2/5, 1/3, 2/3) : -16, -12, -10, -8, -4, -2, 0, 4$
 $10_{68}^{*\dagger} = K(3/5, 1/3, 1/3) : -14, -10, -8, -6, -4, -2, 0, 2, 4, 6$
 $10_{69} = K(3/5, 2/3, 2/3) : -6, -2, 0, 2, 4, 6, 8, 10, 12, 14$
 $10_{70} = K(2/5, 1/3, 3/2) : -8, -4, -2, 0, 2, 4, 6, 8, 10, 12$
 $10_{71} = K(2/5, 2/3, 3/2) : -10, -6, -4, -2, 0, 2, 4, 6, 8, 10$
 $10_{72} = K(3/5, 1/3, 3/2) : -4, 0, 2, 4, 6, 8, 10, 12, 14, 16$
 $10_{73} = K(3/5, 2/3, 3/2) : -14, -10, -8, -6, -4, -2, 0, 2, 4, 6$
 $10_{74} = K(1/3, 1/3, 5/3) : -16, -10, -4, 0, 4$
 $10_{75} = K(2/3, 2/3, 5/3) : -8, -2, 0, 4, 8, 12$
 $10_{76} = K(1/3, 1/3, 5/2) : -4, 0, 2, 6, 8, 12, 14, 16$
 $10_{77} = K(1/3, 2/3, 5/2) : -6, -2, 0, 4, 6, 10, 12, 14$
 $10_{78} = K(2/3, 2/3, 5/2) : -16, -12, -10, -6, -4, 0, 2, 4$
 $10_{124} = K(1/5, 1/3, -1/2) : 0, 15$
 $10_{125} = K(1/5, 2/3, -1/2) : -10, 0, 4, 32/5$
 $10_{126} = K(4/5, 1/3, -1/2) : -14, -8, -6, -4, 0, 8/3$
 $10_{127} = K(4/5, 2/3, -1/2) : -16, -10, -8, -6, -2, 0, 1$
 $10_{128} = K(3/7, 1/3, -1/2) : 0, 8, 32/3, 16$
 $10_{129}^* = K(3/7, 2/3, -1/2) : -10, -2, 0, 1, 4, 20/3$
 $10_{130} = K(4/7, 1/3, -1/2) : -14, -8, -4, 0, 8/3$
 $10_{131} = K(4/7, 2/3, -1/2) : -16, -10, -6, -2, 0, 1$
 $10_{132}^* = K(2/7, 1/3, -1/2) : -14, -2, 0, 3/2$
 $10_{133} = K(2/7, 2/3, -1/2) : -16, -6, -10/3, -2, 0, 1/2$
 $10_{134} = K(5/7, 1/3, -1/2) : 0, 4, 6, 8, 10, 14, 50/3$
 $10_{135} = K(5/7, 2/3, -1/2) : -10, -6, -4, -2, 0, 4, 6, 7$
 $10_{136}^* = K(2/5, 2/5, -1/2) : -8, 0, 1, 14/3, 8$
 $10_{137} = K(2/5, 3/5, -1/2) : -12, -8, -4, 0, 1, 14/3$
 $10_{138}^* = K(3/5, 3/5, -1/2) : -8, -4, -2, 0, 4, 8, 9$
 $10_{139}^{*\dagger} = K(1/4, 1/3, -2/3) : 0, 12, 13, 18, 20$

$$\begin{aligned}
10_{140} &= K(1/4, 1/3, -1/3) : -14, 0, 8/5 \\
10_{141}^{*\dagger} &= K(1/4, 2/3, -1/3) : -12, -4, -2, 0, 2, 9/2 \\
10_{142} &= K(3/4, 1/3, -2/3) : 0, 8, 12, 16 \\
10_{143}^{*\dagger} &= K(3/4, 1/3, -1/3) : -14, -8, -6, -2, 0, 8/3 \\
10_{144} &= K(3/4, 2/3, -1/3) : -12, -8, -4, -2, 0, 2, 5 \\
10_{145}^{*\dagger} &= K(2/5, 1/3, -2/3) : -18, -6, -4, 0, 2 \\
10_{146}^{*\dagger} &= K(2/5, 2/3, -1/3) : -10, -4, -2, 0, 2, 3, 4, 20/3 \\
10_{147}^{*\dagger} &= K(3/5, 1/3, -1/3) : -8, -4, -2, 0, 4, 6, 26/3
\end{aligned}$$

The program computes the Euler characteristic and number of boundary components of the “simplest” surface for each boundary slope. In the above knots, the only genus zero incompressible surfaces are the annuli in the torus knots $3_1, 5_1, 7_1, 8_{19}, 9_1$, and 10_{124} . The non-two-bridge Montesinos knots above with boundary slopes realized by genus one incompressible surfaces are:

$$\begin{aligned}
8_5 &: 12 \\
8_{20} &: 0 \text{ (non-Seifert surface)} \\
9_{35} &: 0 \text{ (Seifert surface)} \\
9_{42} &: 6 \\
9_{46} &: 2, 0 \text{ (Seifert surface)} \\
10_{46} &: 16 \\
10_{61} &: 12 \\
10_{125} &: 4 \\
10_{126} &: -4 \\
10_{132}^* &: -2 \\
10_{139} &: 12, 13 \\
10_{140} &: 0 \text{ (non-Seifert surface)} \\
10_{142} &: 12 \\
10_{145} &: -6, -4
\end{aligned}$$

Some more complicated examples:

$$\begin{aligned}
K(2/5, 3/7, -1/3, -5/8)^* &: -14, -10, -8, -6, -16/3, -4, \\
&\quad -23/6, -2, -8/23, 0, 2, 4, 6, 8, 10, 12, 14, 16, 20, \infty \\
K(2/3, 1/3, -3/5, -3/4, 3/7) &: -18, -14, -12, -10, -8, \\
&\quad -6, -4, -2, -4/3, 0, 1/2, 80/51, 2, 24/7, 38/11, 4, \\
&\quad 16/3, 6, 8, 10, 12, 14, 16, 20, \infty \\
K(1/3, 1/3, -1/3, -2/5, 1/5, -3/4, 2/3) &: -16, -12, -10, \\
&\quad -8, -6, -4, -2, 0, 2, 4, 6, 8, 10, 58/5, 12, 40/3, 122/9, \\
&\quad 124/9, 14, 102/7, 190/13, 76/5, 168/11, 142/9, 16, \\
&\quad 434/27, 146/9, 18, 20, 22, 24, 26, 28, 30, 34, \infty \\
K(-15/32, 3/11, 7/41)^\dagger &: -34, -30, -28, -26, -24, -22, \\
&\quad -20, -18, -16, -14, -12, -10, -8, -6, -74/15, \\
&\quad -4, -26/7, -2, -13/8, -16/17, -10/11, 0, 2, 44/19, \\
&\quad 40/13, 34/11, 74/21, 4, 6, 86/11, 8, 148/17, 10, 83/8, \\
&\quad 152/13, 12, 127/10, 216/17, 14, 272/19, 16, 167/10, \\
&\quad 18, 20, 22, 24 \\
K(11/53, 17/43, -13/21) &: -36, -32, -28, -24, -47/2, \\
&\quad -22, -62/3, -20, -39/2, -18, -390/23, -50/3, \\
&\quad -16, -44/3, -594/41, -72/5, -14, -68/5, -12, -10, \\
&\quad -48/5, -8, -13/2, -6, -28/5, -26/5, -14/3, -22/5, \\
&\quad -4, -5/2, -2, -6/5, -2/3, -2/5, 0, 2, 3, 7/2, 4, 24/5,
\end{aligned}$$

$$6, 15/2, 8, 44/5, 10, 23/2, 12, 38/3, 64/5, 14, 16, 50/3, 18, 20, 22, 24$$

$$\begin{aligned}
K(1/3, 3/5, -3/4, -2/7, 3/11, -5/13)^* &: -14, -10, -8, \\
&\quad -6, -4, -2, 0, 2, 4, 664/117, 6, 20/3, 38/5, 8, 62/7, \\
&\quad 9, 19/2, 776/81, 48/5, 260/27, 10, 32/3, 98/9, 11, \\
&\quad 58/5, 82/7, 12, 110/9, 112/9, 90/7, 13, 27/2, 122/9, \\
&\quad 68/5, 96/7, 14, 72/5, 44/3, 134/9, 15, 46/3, 108/7, \\
&\quad 78/5, 110/7, 16, 146/9, 148/9, 118/7, 17, 52/3, 35/2, \\
&\quad 88/5, 230/13, 124/7, 18, 92/5, 170/9, 96/5, 58/3, \\
&\quad 136/7, 138/7, 20, 106/5, 64/3, 43/2, 282/13, 152/7, \\
&\quad 22, 116/5, 24, 276/11, 26, 28, 30, 32, 34, 36, 38, 42, \infty
\end{aligned}$$

3 Preparing the table

In this section I briefly describe the precautions taken to insure that the table in Section 2 is correct. I first noticed errors in the table in [HO] when comparing it with the output of a program that computes the *normal* boundary slopes of a knot (that is, the boundary slopes of all surfaces which are normal with respect to a choice of triangulation of the exterior of the knot). The data in Section 2 is consistent with the normal boundary slope data. Following the algorithm given in [HO], but without reference to the program used in computing the table there, I wrote a completely new program to compute boundary slopes. I then compared its output with the output of Hatcher and Oertel’s program. When the output differed, I debugged both programs until I found the source of the problem, and then fixed the appropriate program. Eventually, after the new program and the fixed version of Hatcher and Oertel’s program had agreed for thousands of trial Montesinos knots, I declared victory and went home.

Unfortunately, there remained several cases where I implemented the algorithm incorrectly and Hatcher and Oertel’s program also made a very similar mistake. Comparing the 1999 version of this note with Marc Culler’s computations of A -polynomials [C], Thomas Mattman found some additional errors, leading to this 2010 revision. For the table of knots with at most 10 crossings, the 1999 version of this note is strictly better than [HO], i.e. some errors were corrected but no new ones were introduced. Hopefully, the tables are now completely correct, but it is possible that errors remain.

One source of problems in the table in [HO] is a slight error in the main body of [HO]. The remark immediately preceding Proposition 2.7 gives conditions that are claimed to be equivalent to certain hypotheses of Propositions 2.6 and 2.7. It was this reformulation of these propositions which was used in Hatcher and Oertel’s program. However, the conditions given in the remark are not in fact equivalent to those given in the propositions.

My program used the original formulation.

4 Getting the program

The programs I used in preparing this note are available at <http://dunfield.info/montesinos>. They are written the programming language Python, and you will need a Python interpreter to run them. These interpreters are available, for free and for almost all platforms, from <http://python.org>.

References

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